On the Relationship Between Intuitionistic Propositional Logic and Intuitionistic Protothetics

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This work was partially supported by RFBR grant No 98–06–80205 "Logical theory of informalizability".

Let **Int** be the intuitionistic propositional logic, Var be a set of propositional variables and $\mathbf{T}$ be a theory formulated in some rich language $\mathcal{L}$. One may conceive either the 1-st or 2-nd order language, or the propositional language with propositional quantification, etc.

The language $\mathcal{L}$ is assumed to contain the standard intuitionistic connectives $\to$, $\vee$, $\wedge$, $\neg$. Also we suppose that the class of all $\mathcal{L}$-formulas includes the subclass $Sent(\mathcal{L})$ of *sentences*.

A function $t : Var \to Sent(\mathcal{L})$ is said to be an *interpretation*. An arbitrary interpretation is expanded on the class of all propositional formulas by means of induction:

$t(A \circ B) = t(A) \circ t(B)$ for binary connective $\circ$, and $t(\neg A) = \neg t(A)$.

Let the inference system of $\mathbf{T}$ supports the standard axiom schemata and rules of **Int**. Then the following statement is undoubtful:

"If $\mathbf{Int} \vdash A$*, then for every interpretation* t*,* $\mathbf{T} \vdash t(A)$*".*

(*the soundness theorem*). The inverse statement

"If $\mathbf{Int} \nvdash A$*, then for some interpretation* t*,* $\mathbf{T} \nvdash t(A)$*"*

one may treate as *the completeness theorem* for **Int** w.r.t. $\mathbf{T}$.

The completeness of **Int** w.r.t. the intuitionistic arithmetic HA (Heyting arithmetic) is known as *de-Jongh–Smorynski theorem.*

We present a similar result for *quite intuitionistic prototetics* $\mathbf{Int2_i}$, i.e. for intuitionistic propositional logic with propositional quantification, with an additional axiom $\neg\forall p\,(p \vee \neg p)$ (the negation of *tertium non datur*).

Theorem. *Let* $A(p_1, \ldots, p_n)$ *be a propositional formula, such that* $\mathbf{Int} \nvdash A(p_1, \ldots, p_n)$*. Then there are sentences* $E_1, \ldots, E_n$ *of the language of protothetics, such that* $\mathbf{Int2_i} \nvdash A(p_1|E_1, \ldots, p_n|E_n)$*.*

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