Extending the Cut-Free Version of the Second Incompleteness Theorem Almost to Robinson's Arithmetic Q

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Let us recall that Robinson's Arithmetic system Q differs from Peano Arithmetic by containing no Induction axioms [3]. In 1985, Pudlák strengthened a theorem by Bezboruah and Shepherdson [2] to show that the *cut-permissive version* of the Second Incompleteness (that discussed the capacities of the Hilbert-style deduction calculus) generalized from Peano Arithmetic to **all possible extensions** of Q.

The main gap left by the prior literature was it had not resolved to what extent the Second Incompleteness Theorem was valid for weak systems with respect to cut-free deductive methods. In particular, Adamowicz [1]) had shown that $I\Sigma_0 + \Omega_2$ satisfied the Second Incompleteness property for semantic tableaux deduction, but it was not known how much closer to Robinson's Arithmetic Q this result would extend.

We have obtained a partial (albeit not complete) answer to this question. Let d denote any cut-free method of deduction, such as for instance the Herbrand methodology, Semantic Tableaux, Resolution or the Cut-Free Sequent Calculus. For each such d , we have proven there exists a corresponding Π_1 sentence, called V_d , such that if α is any consistent extension of $Q + V_d$ then α cannot prove that the deduction-method d applied to the axiom set α is assured not to produce a proof of 0=1.

Our prior research [5, 6, 7] had documented the existence of some exceptions to the cut-free version of the Second Incompleteness Theorem when an axiom system had a capacity to prove only Addition is a total function (and it viewed Multiplication instead as a 3-way relation). The combination of this prior negative result and the new positive results, reported in the preceding paragraph, tightly complement each other.

[1] Z. Adamowicz, On The Tableaux consistency in weak theories , circulating manuscript from the Mathematics Institute of the Polish Academy of Sciences (1999).

[2] A. Bezboruah & J. Shepherdson, Gödel's Second Incompleteness Theorem for Q , JSL 41 (1976), pp. 503–512.

[3] E. Mendelson, Introduction to Math Logic, Wadsworth-Brooks-Cole Math Series, 1987. See page 157 for Q's definition.

[4] P. Pudlák, Cuts Consistency Statements and Interpretations, JSL 50 (1985), pp. 423–442.

[5] D. Willard, Self-Verifying Axiom Systems, Third Kurt Gödel Symp (1993), Springer-Verlag LNCS#713, pp. 325–336.

[6] D. Willard, The Tangibility Reflection Principle, Fifth Kurt Gödel Symp (1997) SpringerVerlag LNCS#1289, pp. 319-334.

[7] D. Willard, Self-Verifying Systems, the Incompleteness Theorem & Tangibility Reflection Principle, to appear in the JSL.

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