Winning Strategies and σ-closed Sets of Partial Isomorphisms

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Let I_κ be the set of all partial isomorphisms of cardinality $< \kappa$ between two first order structures $\mathbf{A}$ and $\mathbf{B}$ of the same vocabulary. $X \subseteq I$ has the κ-back-and-forth property if for all $\lambda < \kappa$ and for all $p \in X$: $\forall a \in {}^\lambda A \exists b \in {}^\lambda B\ (p \cup \{\langle a(i), b(i)\rangle : i < \lambda\} \in X)$ and $\forall b \in {}^\lambda B \exists a \in {}^\lambda A (p \cup \{\langle a(i), b(i)\rangle : i < \lambda\} \in X)$. X is σ-closed, if whenever $p_0 \subseteq p_1 \subseteq \ldots$ are in X, also $\bigcup_{n<\omega} p_n \in X$. Let $I^*_{\aleph_1}$ be the largest $X \subseteq I$ with the $\aleph_1$-back-and-forth property. If $I^*_{\aleph_1} \neq \emptyset$, then $\mathbf{A} \equiv_{\infty\aleph_1} \mathbf{B}$. The following theorem is a special case of a more general result:

Theorem 1 *The following conditions are equivalent:*

(1) *The Nonempty Player has a winning strategy in the Banach-Mazur game on* $\langle I^*_{\aleph_1}, \subseteq\rangle$.

(2) $I^*_{\aleph_1}$ *contains a σ-closed dense subset.*

Another measure of the largeness of I^*_λ is the Ehrenfeucht-Fraïssé-game.

Theorem 2 *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *have cardinality* $\leq \lambda = \kappa^+$. *Then the following conditions are equivalent:*

(3) *Player II has a winning strategy in the Ehrenfeucht-Fraïssé-game of length* ω_1 *between* $\mathbf{A}$ *and* $\mathbf{B}$, *players playing* $\leq \kappa$ *elements at a time.*

(4) I^*_λ *has a σ-closed subset with the λ-back-and-forth property.*

The condition (4) was introduced by Dickmann in 1975. It implies isomorphism among structures of cardinality $\leq \aleph_1$. It is an open problem, whether (4) is transitive for all $\mathbf{A}$ and $\mathbf{B}$.

Corollary 3 (4) *is transitive among structures of cardinality* $\leq \lambda = \kappa^+$.

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