Canonical Behaviour of Borel Functions on Polish Planes

Otmar Spinas

A subset of the Baire space ${}^\omega\omega$ is called *superperfect* if it is closed and homeomorphic to ${}^\omega\omega$. A rectangle in $({}^\omega\omega)^2$ is called *superperfect* if both sides are superperfect. I will present the canonization of all Borel measurable functions from $({}^\omega\omega)^2$ to $\mathbb{R}$ modulo restriction to superperfect rectangles. By Hurewicz' Theorem this yields, for every Polish space X which is not σ-compact, the canonization of all Borel functions $f : X^2 \to \mathbb{R}$ modulo restriction to rectangles with both sides closed and not σ-compact. It is known that an analogous result is impossible in dimension 3 and higher. Below I explain how the canonization looks like. This is a continuation of the work in [Sp1] and is presented in [Sp2].

Definition 1. We say that $(x, y) \in ({}^\omega\omega)^2$ *oscillates infinitely often* if there exists a strictly increasing sequence $\langle k_i : i < \emptyset \rangle \in {}^\omega\omega$ such that $k_0 = 0$ and the following hold for all $n < \omega$:

1. $k_{2n} = \min\{i < \omega : y(i) > \max\{k_{2n+1}, x(j) : j < k_{2n+1}\}\}$,
2. $k_{2n+1} = \min\{i < \omega : x(i) > \max\{k_{2n+2}, y(j) : j < k_{2n+2}\}\}$.

As $\langle k_i : i < \omega \rangle$ is uniquely determined, we let $\mu_0(x, y) = \emptyset$, $\mu_{n+1}(x, y) = x \restriction k_{2n+1}$, $\nu_n(x, y) = y \restriction k_{2n}$, for all $n < \omega$. Let IO denote the set of all $(x, y) \in ({}^\omega\omega)^2$ which oscillate infinitely often. Let ETP (the set of *even type-pairs*) denote the set of all pairs of the form $(\mu_n(x, y), \nu_n(x, y))$, for some $(x, y) \in$ IO and $n < \omega$. Similarly, OTP denotes the set of all *odd type-pairs*, i.e. those of the form $(\mu_{n+1}(x, y), \nu_n(x, y))$, for some $(x, y) \in$ IO and $n < \omega$.

Definition 2. Let c :ETP$\to 2$, d :OTP$\to 2$ be arbitrary. Define functions f_c and f_d with domain IO as follows: Fix $(x, y) \in$ IO and let $(\mu_n, \nu_n) = (\mu_n(x, y), \nu_n(x, y))$. If $\langle(\mu_{n_k}, \nu_{n_k}) : k < N \leq \omega\rangle$ is the subsequence of those pairs (μ_n, ν_n) with $c(\mu_n, \nu_n) = 1$, then let $f_c(x, y) = \langle \mu_{n_k+1} : k < N\rangle$. Similarly, if $\langle(\mu_{n_k+1}, \nu_{n_k}) : k < N \leq \omega\rangle$ is the subsequence of those (μ_{n+1}, ν_n) with $d(\mu_{n+1}, \nu_n) = 1$, let $f_d(x, y) = \langle \nu_{n_k+1} : k < N\rangle$.

Theorem. *Suppose that $f : ({}^\omega\omega)^2 \to \mathbb{R}$ is Borel measurable. There exists a superperfect rectangle $S_0 \times S_1 \subseteq IO$ such that precisely one of the following alternatives holds:*

1. $\forall (x, y), (x', y') \in S_0 \times S_1 \quad f(x, y) = f(x', y')$.
2. $\forall (x, y), (x', y') \in S_0 \times S_1 \quad f(x, y) = f(x', y') \Leftrightarrow x = x'$.
3. $\forall (x, y), (x', y') \in S_0 \times S_1 \quad f(x, y) = f(x', y') \Leftrightarrow y = y'$.
4. *There exists c :ETP$\to 2$ such that for all $(x, y), (x', y') \in S_0 \times S_1$ we have $c(\mu_n(x, y), \nu_n(x, y)) = 1$ for infinitely many n and $f(x, y) = f(x', y') \Leftrightarrow f_c(x, y) = f_c(x', y')$.*
5. *There exists d :OTP$\to 2$ such that for all $(x, y), (x', y') \in S_0 \times S_1$ we have $c(\mu_{n+1}(x, y), \nu_n(x, y)) = 1$ for infinitely many n and $f(x, y) = f(x', y') \Leftrightarrow f_d(x, y) = f_d(x', y')$.*
6. $\forall (x, y), (x', y') \in S_0 \times S_1 \quad f(x, y) = f(x', y') \Leftrightarrow (x, y) = (x', y')$.

[Sp1] **O. Spinas,** *Ramsey and freeness properties of Polish planes*, **Proc. London Math. Soc.**, to appear.

[Sp2] **O. Spinas,** *Canonical behaviour of Borel functions on superperfect rectangles*, preprint.

Address: Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel
Ludewig-Meyn-Strasse 4, 24098 Kiel, Germany
E-mail: `spinas@math.uni-kiel.de`