Large Branches of the E-tree and Jump Pseudo-hierarchies

V. Yu. Shavrukov

Let Σ be the lattice of Σ_1 sentences modulo provability in Peano arithmetic. Thus the largest element $\top$ of Σ is represented by the sentence $0 = 0$.

The *E-tree* is the collection of proper prime filters on Σ partially ordered by inclusion. As the name suggests, this poset is tree-like in the sense that the predecessors of any element are linearly ordered. $\{\top\}$ is the root of the E-tree.

We study branches through the E-tree. It turns out that Turing complexity properties of prime filters hold the key to the structure of maximal branches.

A *jump pseudo-hierarchy (j.p.h.)* is a triple $(P, \leq, (X_p)_{p\in P})$ where $P, X_p \in \omega$, and $\leq$ is a linear order on P such that the sets

$$\{\, (q,r) \in P^2 \mid q \leq r < p \,\} \quad \text{and} \quad \{\, (q,m) \in P \times \omega \mid q < p \text{ and } m \in X'_q \,\}$$

are recursive in X_p, uniformly in p. ($'$ is the Turing jump).

A *coloured j.p.h.* is a j.p.h. $(P, \leq, (X_p)_{p\in P})$ together with a colouring $c : P \to 5$ of P in 5 colours such that the set $\{\, (q, c(q)) \mid q < p \,\}$ is recursive in X_p, uniformly in p.

Our main theorem characterizes the possible order types of branches through the E-tree as those coded by coloured j.p.h.'s satisfying a simple structural condition (or, more precisely, by the coloured order types admitting a j.p.h.).

The coding of order types by pseudo-hierarchies is such that the coded order type "resembles" the order $\leq$ of the pseudo-hierarchy.

Further, the possible order types of (coloured) j.p.h.'s are described by the formula

$$\Big(\sum_{\iota<\gamma} \lambda_\iota \times \varpi_\iota\Big) + \beta,$$

where

— $\times$ is the lexicographic product,

— $(\lambda_\iota)_{\iota<\gamma}$ is any strictly increasing countable sequence of countable ordinals $> \omega$ such that λ_ι is admissible relative to $\{\lambda_\kappa\}_{\kappa<\iota}$,

— $\varpi_\iota = \begin{cases} 1+\eta & \text{if } \lambda_\iota > \sup_{\kappa<\iota} \lambda_\kappa \\ \eta & \text{if } \lambda_\iota = \sup_{\kappa<\iota} \lambda_\kappa, \end{cases}$

— η is the dense countable order type without end points, and

— β is any countable ordinal.

Address: Department of Mathematics and Computer Science, University of Leicester, Leicester LE1 7RH, U.K.
E-mail: `volodya@mcs.le.ac.uk`