A Generalized Realizability for Constructive Arithmetics

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We introduce a generalized notion of realizability, appropriate for constructive arithmetics, in particular Basic Arithmetic, **BA**. This arithmetic is equivalent of **HA** on intuitionistic logic and of **PA** on classical logic, based on Basic Logic, the propositional counterpart of which, was invented by Visser, and is developed for the quantifier case by Ardeshir and Ruitenburg [1].

The crucial different cases from the Kleene's one, are $\to$ and $\forall$. We restrict the recursive functions, associated for these cases, to a definable subclass, for example total recursives or primitive recursives. So the primitive recursive (PR-)realizability for $A \to B$ is:

$$n \ \mathbf{r}^{PR} A \to B \ \equiv \ PR(n) \wedge \forall x (x \ \mathbf{r}^{PR} A \longrightarrow \exists z \{\mathbf{T}(n,x,z) \wedge \mathbf{U}z \ \mathbf{r}^{PR} B\})$$

The (primitive recursive) formula $PR(x)$, intuitively says that in the "program" of the recursive function with code x, there is no use of (unbounded) minimalization. So if $\mathbb{N} \models PR(n)$ then φ_n is primitive recursive (but not necessary vice versa.)

One can prove the soundness of PR-realizability for **BA**, in the usual way. In the "usual" proof for **HA**, one can not restrict the partial recursives to a proper subclass, so it seems that Kleene's definition, for HA, can not be optimized in this way.

A generalized form of *Church's Thesis*, $\mathbf{GCT^{PR}}$:

$$\forall x \Big(\exists z (z \ \mathbf{r}^{PR} A(x)) \to \exists y \phi(x,y) \Big) \Rightarrow \exists u [PR(u) \wedge \forall x \Big(\exists z (z \ \mathbf{r}^{PR} A(x)) \to \exists w \{\mathbf{T}(u,x,w) \wedge \phi(x, \mathbf{U}w)\} \Big)]$$

is realizable in **BA**, and the completeness theorem of Troelstra [2], holds for $\mathbf{BA} + \mathbf{GCT}^{PR}$ and $\mathbf{r}^{PR}$:

$$\mathbf{BA} \vdash \exists x (x \ \mathbf{r}^{PR} \psi) \iff \mathbf{BA} + \mathbf{GCT^{PR}} \vdash \psi$$

As applications, we deduce that **BA** enjoys the traditional constructive properties: Disjunction Property and Explicit Definability by Numbers, also each provably total function of **BA** is primitive recursive.

Keywords: Realizability, Basic Logic, Basic Arithmetic.

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[1] Ruitenburg, Wim; *Basic Predicate Calculus*, Notre Dame J. of Formal Logic, Vol.**39**, N.1, (1998) pp. 18-46

[2] Troelstra, A.S.(ed.); **Metamathematical Investigations of Intuitionistic Arithmetic and Analysis**, Springer-Verlag, Berlin 1974.

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