Model-theoretic Differential Geometry

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A *differential field* is here a structure $(k, \partial_0, \ldots, \partial_{m-1})$, where k is a field of characteristic zero, and the unary functions ∂_i are commuting derivations. The model-completion of the theory of differential fields is axiomatized by the Frobenius Theorem of differential geometry. Similar arguments go through formally in case the derivations span a non-commutative Lie algebra.

The Frobenius Theorem is usually stated in terms of a real manifold, but it has a purely algebraic content. If K is a field-extension of k, then the possible extensions of the derivations ∂_i to K span a vector space, whose dual may be called the space of *differential* 1*-forms of* K *over* k. There are 1-forms $\mathrm{d}T^i$, dual to the derivations ∂_i. A system of 1-forms ω of K has a *generic solution* if and only if the 2-forms $\mathrm{d}\omega$ are in the space generated by the forms ω, *and* the forms $\mathrm{d}T^i$ are linearly disjoint from the forms ω. (That the system has a generic solution means that the kernel of the forms ω is spanned by particular *commuting* extensions of the derivations ∂_i)

Tracey McGrail ('The model theory of partial differential fields', *J. Symbolic Logic*, to appear) and Yoav Yaffe ('Model completion of Lie differential fields', Master's Thesis, Hebrew University, 1998) have shown independently that the theory of differential fields has a model-completion, so their axioms must be equivalent to the Frobenius Theorem; but the latter can also be translated directly into an infinite set of first-order axioms.

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