On Non-Wellfounded Sets as Fixed Points of Substitutions

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(Joint work with Tapani Hyttinen)

We work in set theory with urelements, denoted by $\mathcal{U}$, and without the foundation axiom, ZFC^-. In [1] non-wellfounded sets and the antifoundation axiom, AFA are studied. We denote by $V_{afa}[\mathcal{U}] \cup \mathcal{U}$ the model of $ZFC^- + AFA$. Substitution operations $sub(s, b)$, also denoted by $b[s]$, are studied in [1]. Here s is a function, called substitution, such that $dom(s) \subseteq \mathcal{U}$, and $b \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$, such that the following conditions hold: (i) If $x \in dom(s)$, then $sub(s, x) = s(x)$. (ii) If $x \in \mathcal{U} - dom(s)$, then $sub(s, x) = x$. (iii) For all sets b, $sub(s, b) = \{sub(s, p) \mid p \in b\}$. There is a unique substitution operation $sub(s, b)$ defined for all substitutions s and $b \in V_{afa}[\mathcal{U}] \cup \mathcal{U}$.

Substitutions can be composed in the following way. A substitution s is proper if for all $x \in dom(s)$, $s(x) \in V_{afa}[\mathcal{U}]$ whenever $s(x) \neq x$. If s and t are substitutions, then $t \star s$ is the substitution whose domain is $dom(s)$ and for every $x \in dom(s)$, $(t \star s)(x) = s(x)[t]$. Now AFA is equivalent to: for every proper substitution e there is a unique proper substitution s such that $s = s \star e$.

We study the non-wellfounded sets as fixed points of substitutions. If f is a function, let f^* be the class function such that if $x \in dom(f)$, $f^*(x) = f(x)$, otherwise $f^*(x) = x$. If f and g are functions, let $f[g]$ be the function such that $dom(f[g]) = dom(g)$ and $f[g](x) = f^*(x)$, if x is an urelement, and otherwise $f[g](x) = \{g^*(y) \mid y \in f^*(x)\}$. Note that if $f(x)$ is a set, then $g(x) = \{g(y) \mid y \in f(x)\}$. We say that g is a *fixed point* of f, $(FP(g, f))$, if $dom(f) \cup \bigcup f^*[dom(g)] \subseteq dom(g)$ and $g = f[g]$. The *fixed point axiom, FA* is: every function has a fixed point. We have shown that assuming the strong extensionality axiom, FA and AFA are equivalent. In ZFC, let X be a set and $f : X \to \mathcal{P}(X)$, be such that $f(x) = x \cap X$. Then f has a unique fixed point g, namely the Mostowski collapse of X. We will show that from the FA it follows that for every substitution f there is g such that $g = g \star f$. However, g is not necessarily unique, and for arbitrary f, g does not necessarily exist.

[1] Jon Barwise and Lawrence Moss, **Vicious Circles**, CSLI Lecture Notes, Number 60, CSLI Publications, Stanford, 1996.

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