An Application of Ramsey's Theorem to Groups with Homogeneous Theory

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We investigate the following conjecture,which was proposed by H. Kikyo in 1996:

Definition. For any integers $n > m \geq 2$, a complete theory T is (m, n)-*homogeneous* if, for each model M of T, two n-tuples $\overline{a}, \overline{b}$ in M have the same type if the corresponding m-tuples from $\overline{a}$ and $\overline{b}$ have the same type.

Conjecture. If M is an infinite group, with possibly additional structure, then, for any integers $n > m \geq 2$, the theory of M is not (m, n)-homogeneous.

We proved the theorem below, using a variant of Ramsey's Theorem which follows from more general results by R.L. Graham, K. Leeb and B.L. Rotschild:

Theorem. Let $\mathcal{L}$ be a language, let M be an $\mathcal{L}$-structure, and let $*(u, v, w)$ be a parameter-free $\mathcal{L}$-formula. If $*$ defines an abelian group structure on an infinite subset or on arbitrarily large finite subsets of M, then the theory of M is not (m, n)-homogeneous for $n > m \geq 2$.

Consequently, if M is a counterexample to Kikyo's conjecture, then there exists an integer h such that each abelian subgroup of M has at most h elements. It follows that there exist an integer k such that $M^k = 1$, and an integer l such that each finite subgroup of M has at most l elements. The group M necessarily has infinite finitely generated subgroups, and any such subgroup is a counterexample to Burnside's conjecture.

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