# Some Restricted Lindenbaum Theorems Equivalent to the Axiom of Choice

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$\langle S, \mathbf{Cn}\rangle$ is a *deductive calculus* if $\mathbf{Cn}$ is a function from $\wp(S)$ to $\wp(S)$ for which (a) if $X \subseteq S$ then $X \subseteq \mathbf{Cn}(X) \subseteq S$; (b) if $X \subseteq S$ then $\mathbf{Cn}(\mathbf{Cn}(X)) = \mathbf{Cn}(X)$; and (c) if $X \subseteq S$ then $\mathbf{Cn}(X) = \bigcup\{\mathbf{Cn}(Z) \mid Z \subseteq X \text{ and } |Z| < \aleph_0\}$. A set $Y \subseteq S$ is a *theory* in $\langle S, \mathbf{Cn}\rangle$ if $Y = \mathbf{Cn}(Y)$; a theory $\mathbf{Y}$ is *$y$-saturated* in $\langle S, \mathbf{Cn}\rangle$ if (a) $y \notin \mathbf{Y}$ and (b) if $x \notin \mathbf{Y}$ then $y \in \mathbf{Cn}(\mathbf{Y} \cup \{x\})$. The generalized Lindenbaum extension theorem *LET* (for whose proof the axiom of choice *AC* is needed) says that if $y \notin \mathbf{Y}$ then $\mathbf{Y}$ has a $y$-saturated extension.

Dzik [1981] gives a direct proof of *AC* from a premise *let* that attributes only a weak version of the Lindenbaum property, namely the existence of one saturated theory, to a restricted set of deductive calculi (in particular $\mathbf{Cn}(Y)$ takes only the values $Y$ and $S$). Within ZF, plainly, *let* and *LET* are equivalent to *AC*. The question arises whether the conditional *let* $\Rightarrow$ *LET* can be proved in ZF without in effect first proving *AC* from *let*, and then proving *LET* from *AC*. The paper provides a partial answer to this question.

Dzik, W. [1981]. 'The Existence of Lindenbaum's Extensions is Equivalent to the Axiom of Choice'. *Reports on Mathematical Logic* **12**, pp. 29-31.

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