Intuitionistic Logic and Projective Beth Property

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We consider the family $E(Int)$ of all superintuitionistic logics. It is well known that there is a duality between $E(Int)$ and the family of varieties of Heyting algebras.

Let $\mathbf{x}$, $\mathbf{q}$, $\mathbf{q}'$ be disjoint lists of variables not containing y and z, $\Gamma(\mathbf{x}, \mathbf{q}, y)$ a set of formulas. We say that a logic L has the *projective Beth property PBP* if
$\Gamma(\mathbf{x}, \mathbf{q}, y), \Gamma(\mathbf{x}, \mathbf{q}', z) \vdash_L y \leftrightarrow z$ implies $\Gamma(\mathbf{x}, \mathbf{q}, y) \vdash_L y \leftrightarrow C(\mathbf{x})$ for some formula C($\mathbf{x}$); the *Beth Property BP* arises from PBP by deleting $\mathbf{q}$ and $\mathbf{q}'$.

It is known [Maksimova 77] that there exist exactly 8 extensions of Int with Craig's interpolation property CIP, and $L \in E(Int)$ has CIP iff its associated variety $V(L)$ has the Amalgamation Property AP. All extensions of Int have BP [Kreisel 60], and CIP implies PBP in $E(Int)$.

We say that a variety V has the property SES if for any $\mathbf{B}$ in V and its subalgebra $\mathbf{A}$, for any $b \in \mathbf{B} - \mathbf{A}$ there exist $\mathbf{C} \in V$ and homomorphisms $\beta : \mathbf{B} \to \mathbf{C}$ and $\gamma : \mathbf{B} \to \mathbf{C}$ such that $\beta(x) = \gamma(x)$ for all $x \in \mathbf{A}$ and $\beta(b) \neq \gamma(b)$.

Lemma 1. *For each logic L in $E(Int)$, L satisfies PBP iff $V(L)$ has SES.*

Proposition 2. *Let L be any logic in $E(Int)$.*
(i) If L has CIP then L and $\Delta(L)$ have PBP.
(ii) If L has PBP then $L + \neg p \vee \neg\neg p$ has PBP.

Here $\Delta(L) = Int + \{p \vee (p \to A) | A \in L$ and p is not in $A\}$ [Hosoi 67]. By this proposition, at least 15 logics in $E(Int)$ have PBP. These logics are finitely axiomatizable and have the finite model property.

Theorem 3. *There exist exactly 16 logics with PBP in $E(Int)$. Their list contains eight logics with CIP and, moreover, five logics of the form $\Delta(L)$ and two logics of the form $\Delta(L) + \neg p \vee \neg\neg p$, where L has CIP, and one more logic which is characterized by four-element linearly ordered Heyting algebra.*

It follows that there are exactly 16 varieties of Heyting algebras with SES.

Theorem 4. *(i) There exist an algorithm which for any extension of Int by finitely many axiom schemes decides whether this extension possesses PBP or not.*
(ii) SES is base-decidable on the family of varieties of Heyting algebras.

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