Valuations with Cyclically Ordered Exponents

Gérard Leloup

(Joint work with M. Giraudet & F.-V. Kuhlmann)

We say that a set C is cyclically ordered if it is equipped with a 3-ary relation $(\cdot,\cdot,\cdot)$ on C that satisfies (1), (2), (3) below.

(1) $(\cdot,\cdot,\cdot)$ enjoys $\forall x,y,z,\ (x,y,z) \Rightarrow x \neq y \neq z \neq x$

(2) $(\cdot,\cdot,\cdot)$ enjoys $\forall x,y,z,\ (x,y,z) \Rightarrow (y,z,x)$

(3) for all $x \in C$, $(x,\cdot,\cdot)$ is antisymetric and transitive.

Let C be an abelian group and a cyclic order $(\cdot,\cdot,\cdot)$, we say that C is a cyclically ordered group if $(\cdot,\cdot,\cdot)$ is compatible, i.e. $\forall x,y,z,u,\ (x,y,z) \Rightarrow (x+u,y+u,z+u)$

Any totally ordered set is a cyclically ordered one once equipped with the ternary relation (x,y,z) iff $x<y<z$ or $y<z<x$ or $z<x<y$. With the same relation, any totally ordered group is a cyclically ordered one. Such a cyclically ordered group is called a linear cyclically ordered one.

Let k be a commutative ring (or field), and C be a cyclically ordered group.
We generalize the usual definitions of $k[[C]]$, the ring of formal power series, and $k[C]$, the ring of polynomials, in such a way that they coincide with classical one when C is a linear cyclically ordered group. We can prove that $k[C]$ is integral if and only if k is integral and C is torsion free, and that $k[[C]]$ is a field if and only if k is a field and C a linear cyclically ordered group. Then, we define the cyclic valuation on $k[[C]]$, and point out some of its properties.
Next, we define cyclically valued groups, we give two first order properties and prove that the subgroups of Hahn products of abelian groups indexed by C are exactly the cyclically valued groups satisfying those properties, and such that the supports of all the elements are well ordered.

If we assume that C is a cyclically ordered group, and extend the definition of cyclically valued group to cyclically valued ring, there exist some properties analogous to the properties of the cyclically valued groups, with formal power series rings with cyclically ordered exponents instead of Hahn products.

Address: UPRESA 7056 (Equipe Logique Paris VII) et Departement de Mathematiques, U.F.R. Sciences, Universite du Maine, avenue Olivier Messiaen, 72085 Le Mans Cedex 9, France
E-mail: leloup@logique.jussieu.fr