Definability of Computable Functionals over the Reals without Equality

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Partially supported by grants for young scientists SB RAS and grants RFBR N 0001-00810, 99-01-00485

Based on an effective theory of continuous domains notions of computability for operators and real-valued functionals defined on the class of continuous functions are introduced. Definability of computable operators and functionals are proposed. We extend ivestigation of computability properties proposed in [3, 4] to the case of the reals without equality. Let $<\mathbf{R}, 0, 1, +, \cdot, <>$ be the standard model of the reals, denoted also by $\mathbf{R}$. For the definitions of continuous domains for the reals I (the interval domains) and notions of computability we refer to the paper [2, 5].

We denote the *dcpo* $\{[u, v]\}$ where $u : [a, b] \to \mathbf{R}$ is lower semucontinuous function, $v : [a, b] \to \mathbf{R}$ is upper semicontinuous function with the natural order as I_f.

We use the definitions of the set of hereditarily finite sets HF($\mathbf{R}$) over a model $\mathbf{R}$, and the definitions of Σ-definability proposed in [1].

THEOREM An operator $F : C[a, b] \to C[a, b]$ is computable if and only if there exists an operator $F^* : I_f \to I_f$ with the following properties:

1. $F^*(\{f\}) = \{F(f)\}$
2. $F^*([u, v]) = [h_1, h_2]$ iff there exist lower-closed Σ-formula ψ_1 and upper-closed Σ-formula ψ_2 such that
$$h_1|_{[x_1,x_2]} > z \; iff \; \mathrm{HF}(\mathbf{R}) \models \psi_1(U_1, U_2, x_1, x_2, z)$$
$$h_2|_{[x_1,x_2]} < z \; iff \; \mathrm{HF}(\mathbf{R}) \models \psi_2(U_1, U_2, x_1, x_2, z)$$
where $U_1(r_1, r_2, c) \rightleftharpoons u|_{[r_1,r_2]} > c$ and $U_2(r_1, r_2, c) \rightleftharpoons u|_{[r_1,r_2]} < c$

COROLLARY A functional $F : C[a, b] \times [a, b] \to \mathbf{R}$ is computable if and only if there exists an operator $F^* : I_f \to I_f$ with the following properties:

1. $F(f, x) = g(x)$ where $F^*(\{f\}) = \{g)\}$
2. $F^*([u, v]) = [h_1, h_2]$ iff there exist lower-closed Σ-formula ψ_1 and upper-closed Σ-formula ψ_2 such that
$$h_1|_{[x_1,x_2]} > z \; iff \; \mathrm{HF}(\mathbf{R}) \models \psi_1(U_1, U_2, x_1, x_2, z)$$
$$h_2|_{[x_1,x_2]} < z \; iff \; \mathrm{HF}(\mathbf{R}) \models \psi_2(U_1, U_2, x_1, x_2, z)$$
where $U_1(r_1, r_2, c) \rightleftharpoons u|_{[r_1,r_2]} > c$ and $U_2(r_1, r_2, c) \rightleftharpoons u|_{[r_1,r_2]} < c$

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