On Borel Non–reducibility of P-ideals

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Let $\mathcal{E}$ be the least class of equivalence relations on Polish spaces which contains all equalities and is closed under the operations of *countable power* E^∞ and *Fubini product* $\mathrm{Fin} \bigotimes_n \mathsf{E}_n$ over the ideal Fin of all finite sets.

Theorem 1. *Any equivalence relation generated by a Borel P-ideal $\mathcal{I}$ other than* Fin *or* $0 \times \mathrm{Fin}$, *is* <u>*not*</u> *Borel reducible to an equivalence relation in $\mathcal{E}$.*

(Equivalence relations induced by Fin and $0 \times \mathrm{Fin}$ clearly belong to $\mathcal{E}$.)

The subclass $\mathcal{T} \subseteq \mathcal{E}$ obtained by the closure under countable powers only, includes only equivalence relations induced by continuous actions of S_∞, the symmetric group, so that for $\mathcal{T}$ instead of $\mathcal{E}$ Theorem 1 follows from Hjorth's theory of turbulence. (A particular case of $\mathcal{I}$ = the density-0 ideal here, was considered by H. Friedman.) Note that $\mathcal{E}$ contains many remarkable relations which do not belong to $\mathcal{T}$, for instance, *Frechet ideals* Fin_α, obtained from Fin by iteration of Fubini products.

Our technical approach is based on the notion of generic ergodicity, due to Hjorth. An equivalence relation E is *generically* F*-ergodic*, if any Borel map ϑ with $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ is F-constant on a comeager set. We prove that a certain stronger version of the ergodicity is preserved under the same transformations of F as above. Let a Polish group G act continuously on a Polish space X. Say that the induced equivalence E_X^G is *hereditarily generically* (hg, for brevity) F-ergodic if the local relation $\sim_Y^U$ is F-ergodic whenever $U \subseteq G$ is open, symmetric, and contains 1_G, and $\emptyset \neq Y \subseteq X$ is open, and the local orbit $\mathcal{O}(Y, U, x)$ is dense in Y for comeager-many $x \in Y$.

Theorem 2. *Suppose that the action is generically turbulent. Let* F *be a family of analytic equivalence relations such that E_X^G is hg F-ergodic for each F in $\mathcal{F}$. Let $\mathcal{F}'$ be the closure of $\mathcal{F}$ under the operations of countable power and Fubini product over* Fin. *Then E_X^G is hg F'-ergodic for each F' in $\mathcal{F}'$.*

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