Strongly Extensional vs Complementable Collections

Marco Forti

We consider the problem of axiomatizing the primitive notion of *collection*, intended in the most general sense as "any aggregate of distinct objects conceived as a *single* object". In our view, this notion originates from the fusion of two different naive concepts: the "extension of a property" and the "content of a list". The first notion, which can be attributed to Frege, makes implicit reference to a (unique) *universal collection* V: a collection C is obtained by *separating* from V the objects which belong to C (possibly by means of some *definite* property). This concepition yields naturally to postulate *Weak Extensionality* up to urelements, and leaves open the difficult problem of isolating consistent *Comprehension Principles* (Russel's property appears *prima facie* well-defined!). The second notion, more in line with the modern developements of Computer Science, has to define when two lists have the *same* content. A natural choice could be to assume *Strong Extensionality* up to bisimulation (possibly admitting atoms). (This choice is warmly recommended in [1].) Again one has to state suitable *Comprehension Principles*, and now one has to choose between *Singleton* and *Complement*, since V is bisimilar to $V \setminus \{V\}$.

In both cases arbitrary collections cannot be flexible enough to admit the uses of modern Mathematics. Cantor's set theory, later axiomatized by Zermelo, bans large (*absolutely infinite*) collections as inconsistent, and so do the usual axiomatizations *à la* Zermelo. Von Neumann's axiomatization of sets and classes identifies "sethood" with "elementhood", *Limitation of Size* being the basic criterion for both. Hence large collections cannot be collected, and so are intended merely as a "façon de parler". In the spirit of the Foundational Programme of E. De Giorgi, we consider all collections, large and small, as *first class objects.* It seems therefore appropriate to isolate *sets* as "not too large collections", which can be freely manipulated, and introduce the collections $Coll$ (of all collections) and Ins (of all sets). The idea that sets are *small* is embodied in the Axiom of Replacement, while their *simple and controlled internal structure* is rendered by stating that the graph of *membership* between sets and objects is a collection. Strong *Comprehension Principles* are obtained by means of suitable "Gödel Operations" acting on $Coll$: *pairing, union, intersection, cartesian product, domain, etc.*

Adopting the "Fregean" attitude, we should clearly include *complement*, and then we could not assume that the graph of *general membership* is in $Coll$. Hence *unions, intersections,* and *cartesian products* of collections of collections may not exist as collections. This fact would prevent a substantial use of collections in almost all areas of Mathematics, Logic and Semantics. We conjecture that arbitrary unions and intersections are consistent (at least), but in [3] we have only proved the consistency of the weaker axiom: *Unions and intersections of sets of collections and of collections of sets are collections, and so is the cartesian product of a set of collections.*

On the other hand, adopting the "Barwisean" attitude, one can assume that the graph of *general membership* is in $Coll$, thus obtaining the full*Generalized Positive Comprehension GPK* of [2], and in fact a good approximation of the "most wanted" theory SEC asked for in [1]. New problems obviously arise: we cannot enter in details here, and we refer to [2] for a discussion of related topics.

[1] J.BARWISE, L.MOSS - *Vicious Circles.* CSLI L. N. **60**, Stanford 1996.
[2] M.FORTI, F.HONSELL - Choice principles in hyperuniverses. *Ann. Pure Appl. Logic* **77** (1996), 35-52. Addendum and Corrigendum, *ibid.* **92** (1998), 211-214.
[3] M.FORTI, F.HONSELL, M.LENISA - Operations, Collections and Sets within a General Axiomatic Framework, in *Logic and Foundations of Mathematics* (A. Cantini *et al.*, eds.), *Synthese Library* **280**, Kluwer, Dordrecht 1999, 1-24.

Address: Dip. Matematica Applicata "U.Dini", Universita' di Pisa, V. Bonanno 25B - 56100 PISA (Italy)
E-mail: `forti@dma.unipi.it`