$\sum_0^1$- Definitions of Infinity

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Given a model M in a language L, a logic $\mathbf{L}$ (such as first order logic $L_{\omega,\omega}$ or monadic second order logic $L_{\omega,\omega}^{Mon}$) and an infinite cardinal κ, let us say that $\mathbf{L}$ *defines* κ in M if there is an $\mathbf{L}$-formula $\varphi(X)$ in the language $L \cup \{X\}$, where X is a new unary predicate, such that for all subsets A of M, $(M, A) \models \varphi(A)$ iff $\mid A \mid \geq \kappa$. For example, $L_{\omega,\omega}$ does not define $\aleph_0$ in the linear order $(\mathbb{Q} <)$ of the rationals but $L_{\omega,\omega}^{Mon}$ defines $\aleph_0$ in any linear order. However, $L_{\omega,\omega}$ defines $\aleph_0$ in $(\mathbb{R}, <)$, and $L_{\omega,\omega}^{Mon}$ can define $\aleph_1$ in $(\mathbb{R}, <)$, at least in the presence of CH (by an old result of Gurevic). In this talk we survey known results dealing with such definability/undefinability results, and prove some new theorems, such as:

Theorem A: *Let $PA_n = PA$ with induction restricted to $\sum_n$formulas. For each natural number n there is a nonstandard model M of PA_n such that $L_{\omega,\omega}$ defines $\aleph_0$ in M (in contrast to the fact that the only model of PA in which $L_{\omega,\omega}$ can define $\aleph_0$ is the standard model of PA).*

Theorem B: *Suppose F is a real closed field. $L_{\omega,\omega}$ defines $\aleph_0$ in F iff $F \cong \mathbb{R}$ (in contrast with the fact that $L_{\omega,\omega}$ defines $\aleph_0$ in any finite extension of $\mathbb{Q}$).*

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