ZFC[Ω]: a nonstandard set theory where all sets have nonstandard extensions

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Let ZFC[Ω] be the first order theory in the language of set theory plus a binary relation symbol Ω, where all axioms of ZFC except regularity are assumed — separation and replacement schemata are considered for *all* formulas — and the following are postulated. **1)** Ω is a functional correspondance whose domain is the class of all sequences (i.e. functions defined on some infinite cardinal κ). **2)** $\Omega(\varphi) = \{\Omega(\psi) : \psi(i) \in \varphi(i) \text{ for all } i\}$. **3)** If $\vartheta(i) = \{\varphi(i), \psi(i)\}$ for all i, then $\Omega(\vartheta) = \{\Omega(\varphi), \Omega(\psi)\}$. **4)** If $\Omega(\varphi) = \Omega(\psi)$ then $\Omega(f \circ \varphi) = \Omega(f \circ \psi)$ (provided such compositions are defined). For each set A, define its *κ-nonstandard extension* $J_\kappa(A) = \{\Omega(\varphi) : \varphi : \kappa \to A\}$, κ any infinite cardinal. **5)** For every κ, the *κ-nonstandard embedding* J_κ satisfies κ^+-saturation (a set x is internal relative to J_κ if $x \in J_\kappa(A)$ for some A). By restricting to countable sequences, the above axioms can be arranged so to get a truly elementary presentation of nonstandard analysis, suitable even at the freshman level (see [1]).

Theorem. (Transfer principle) *For every κ, J_κ is a nontrivial bounded elementary embedding of the universe into itself.*

In analogy to the von Neumann hierarchy $\bigcup_{\alpha \in ON} V_\alpha(\emptyset)$ of wellfounded sets, define: $W_0 = \emptyset$; $W_{\alpha+1} = \mathcal{P}\big(\bigcup_{\kappa \leq \alpha} J_\kappa(W_\alpha) \cup W_\alpha\big)$; $W_\lambda = \bigcup_{\alpha < \lambda} W_\alpha$ if λ limit. As a substitute of regularity, we postulate: **6)** Every set is in some W_α. In other words, all sets in ZFC[Ω] are obtained by transfinite iterations of power-set and nonstandard extension operations. As a consequence, reflection properties hold.

Theorem. *ZFC[Ω] is equiconsistent with ZFC.*

Symbol Ω is used for historical reasons. In fact, ZFC[Ω] can be presented as a strengthening in the general framework of set theory of the pre-Robinsonian approach to nonstandard analysis by C. Schmieden and D. Laugwitz [2], where Ω denoted a new "infinite" natural number.

[1] V. Benci and M. Di Nasso, *A truly elementary presentation of nonstandard analysis by axiomatic approach*, preprint.

[2] C. Schmieden and D. Laugwitz, *Eine erweiterung der infinitesimalrechnung*, **Math. Zeitschr.**, vol. **69** (1958), pp. 1-39.

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