Model-theoretic Results for Pythagorean Fields

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Notation. (a) A field is *Pythagorean* if every sum of squares is a square. All Pythagorean fields considered here are formally real (= orderable).
(b) For any field F and $n \geq 1$, $I^n(F)$ denotes the n-th power of the ideal of forms of even dimension in the Witt ring of quadratic forms over a field F (modulo Witt-equivalence).
(c) We consider (formally real) fields endowed with their special group (rather than field) language L_{SG}, where addition is replaced by the (much weaker) binary relation

$$a \in D_F(1, b) \text{ iff } F \models (\exists x\, y \in \sum F^2)(a = x + by)$$

($a, b \in F^\times$). The motivation for, and the utility of considering this language can be found in [1]. $G_{red}(F)$ denotes the (multiplicative) group $F^\times / \sum F^{\times 2}$ endowed with this relation and the constant $-1/\sum F^{\times 2}$.
Theorem 1. (Bounds in the theory of Pythagorean fields)
(a) There exists a recursive function $f : N \times N \longrightarrow N$ such that for every $n, m \geq 1$, any Pythagorean field F, and any regular quadratic form φ of dimension m over F, if $\varphi \in I^n(F)$, then φ is a linear combination of at most $f(n, m)$ Pfister forms of degree n.
It follows that membership of $I^n(F)$ can be expressed, uniformly on dimension, by a first-order formula of the language L_{SG} above.
(b) The bound $f(n, m)$ in (a) is simply exponential, of the form m^n with a suitable coefficient, for a very extensive class of Pythagorean fields F (including, f. ex., all those whose associated special group $G(F)$ is locally finite).

Item (a) follows by a compactness argument from our solution to Marshall's problem in quadratic form theory ([2]). Item (b) requires a (rather involved) combinatorial argument.
Theorem 2. (First-order properties of the Pythagorean closure)
(a) For any formally real field F, the special group embedding $G_{red}(F) \hookrightarrow G(F^p)$ induced by the inclusion of F into its Pythagorean closure F^p is not elementary (for the language L_{SG}) whenever the space of orders of F is not perfect.
(b) There are fields F for which the embedding in (a) is not pure.
However,
(c) The morphism in (a) reflects isotropy for all formally real fields F.

These results follow from our Boolean-theoretic methods expounded in [1].

[1] M. Dickmann, F. Miraglia, **Special Groups : Boolean-Theoretic Methods in the Theory of Quadratic Forms**, 247 pp., Memoirs of the AMS 689, May 2000.
[2] M. Dickmann, F. Miraglia, *On forms whose total signature is zero mod 2^n. Solution to a problem of M. Marshall*, Inventiones Math. **133**, 243 - 278 (1998).

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