How a Point-free Constructive Version of the Uryshon Metrization Theorem Can Turn a Collection Into a Set

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In the context of Type Theory the question often arises whether (the collection of points of) a given mathematical space (such as the Continuum, the spaces $\mathcal{L}(\mathcal{A})$ of linear functionals from a semi-normed space A to the reals,...) can be presented as a Set (Data Type) or not. Here we show that some *topological* conditions can be isolated which, if satisfied, allow to give a positive answer. The result is obtained by basing on the framework of Formal Topology (see [Sa87], [Sa99]), here conceived as a constructive and predicative presentation of point-free topology (Locale Theory, see e.g. [J82]) which can be phrased within Per Martin-Löf's Type Theory ([ML84], [NPS90]; see also [SV98]).

We first introduce the notion of *metric formal topology* and prove that it is adequate as a point-free axiomatization of the notion of metric space. A predicative formulation of the topological concept of *complete regularity* is then provided, and the point-free constructive analogue of Uryshon's metrization theorem (stating that a (completely) regular topological space with a countable base is metrizable) is derived.

This metrization result leads to a characterization of the points of (a wide subclass of) compact regular formal topologies with a countable base and, more in general, of locally compact completely regular formal topologies with a countable base: points are shown to coincide with particular subsets of basic neighbourhoods (entirely defined by first-order conditions) in which neighbourhoods of arbitrarily small diameter can be found. It is then easily proved that if a space can be represented via a (locally) compact (completely) regular formal topology with a countable base, the collection of its points can be indexed by a set. We will say that such a space admits a *domain*. In particular, the Continuum, the n-dimensional Euclidean space, Cantor space, and a wide subclass of the class of topologies giving rise to the spaces $\mathcal{L}(\mathcal{A})$ of linear functionals from a semi-normed space A to the reals, admit a domain.

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