Fixed Points for Algorithmic Classes

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Let $\mathbf{A}$ be a class of algorithms over a set X and $\mathbf{E}$ be a set of functions having the form $f : X^m \to X^n$ where n and m are arbitrary natural numbers. Definition 1. A one-to-one mapping $u : \mathbf{A} \to X$ is called a constructive internal naming of $\mathbf{A}$ if u^-1 and u are defined by some algorithms in $\mathbf{A}$. Remark. When $X = N$, then u is called an enumeration. Definition 2. A mapping F:$\mathbf{A} \to \mathbf{E}$ is called a natural projection if for any algorithm A from $\mathbf{A}$ and any x from X we have $F(A(x)) = A(x)$. Definition 3. A function $K(z, x)$ from $\mathbf{E}$ (algorithm $H(z, x)$ from $\mathbf{A}$) is called universal in $\mathbf{E}$ (in $\mathbf{A}$) at the level n if for any $g(x)$ from $\mathbf{E}$ (any $A(x)$ from $\mathbf{A}$) for some fixed $z = a$ and any x from X we have $K(a, x) = g(x)(H(a, x) = A(x))$. Theorem 1. $\mathbf{A}$ has a universal algorithm iff $\mathbf{E}$ has a universal function. Definition 4. A one-to-one mapping $u : X^n \to X$ is called a linearization for the level n. Let us assume that $\mathbf{E}$ contains universal functions and linearization for each level n and all its iterations $w(z(x)) = K(z, x)$ where is a universal algorithm in $\mathbf{A}$. Algorithmic Fixed Point Theorem. For any $A(x_1, ..., x_n + 1)$ from $\mathbf{A}$ there is such $B(x_1, ..., x_n)$ from $\mathbf{A}$ that $w(A(x_1, ..., x_n, B(x_1, ..., x_n))) = w(B(x_1, ..., x_n))$. Corollary 1. For any $A(x)$ from $\mathbf{A}$ there is such a from X that $w(A(a)) = w(a)$. Let $\mathbf{U}$ be some universal class of recursive algorithms. Corollary 2 (Kleene Fixed Point Theorem). For any $A(x_1, ..., x_n + 1)$ from $\mathbf{U}$ there is such $B(x_1, ..., x_n)$ from $\mathbf{U}$ that $w(A(x_1, ..., x_n, B(x_1, ..., x_n))) = w(B(x_1, ..., x_n))$. Let $\mathbf{V}$ be some universal class of recursive algorithms[M. Burgin, *Super-recursive algorithms as a tool for high performance computing*, High Performance Computing, Proceedings, San Diego, 1999]. Corollary 3. For any $A(x_1, ..., x_n+1)$ from $\mathbf{V}$ there is such $B(x_1, ..., x_n)$ from $\mathbf{V}$ that $w(A(x_1, ..., x_n, B(x_1, ..., x_n))) = w(B(x_1, ..., x_n))$.

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