N-Variables Positive Logic and Algebras of Relations

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The theory of algebras of relations plays an important role in modern algebraic logic. Tarski was the first to treat operations on binary relation from an algebraic point of view [1]. In the investigation of algebras of relations, one of the most important problem is to study their laws that can be expressed by equations or conditional equations (see [1-6]).

For any set Ω of operations on binary relations, denote by $R\{\Omega\}$ the class of algebras whose elements are binary relations and whose operations are members of Ω. Let L_∞ be a positive first-order logic [7] and L_n be a restriction of L_∞ on formulas with n individual variables. Denote by $Eq_n\{\Omega\}$ ($Qeq_n\{\Omega\}$) the set of all equations (conditional equations) of algebras from $R\{\Omega\}$ that can be derived in L_n, and let $V_n\{\Omega\}$ ($Q_n\{\Omega\}$) be the corresponding variety (quasivariety).

We shall consider the operations of relation product $\circ$, relation inverse ${}^{-1}$, intersection $\cap$, union $\cup$, and identity relations Δ. It follows from [1] that equations and conditional equations of algebras with these operations can be expressed in L_3. Note that the case $n = 3$ is more or less trivial and we suppose that $n \geq 4$.

The class $R\{\circ, {}^{-1}, \cap, \Delta\}$ was introduced and studied in [3] (see also [5]).

THEOREM 1. $Q_4\{\circ, {}^{-1}, \cap, \Delta\} = V_4\{\circ, {}^{-1}, \cap, \Delta\}$; $V_4\{\circ, {}^{-1}, \cap, \Delta\}$ *is finitely based; an algebra* $(A, \cdot, {}^{-1}, \wedge, e)$ *belongs to* $V_4\{\circ, {}^{-1}, \cap, \Delta\}$ *if and if and only if it satisfies the following equations:* $(xy)z = x(yz)$, $(x^{-1})^{-1} = x$, $(xy)^{-1} = y^{-1}x^{-1}$, $xe = ex = x$, $x \wedge x = x$, $x \wedge y = y \wedge x$, $(x \wedge y) \wedge z = x \wedge (y \wedge = z)$, $(x \wedge y)^{-1} = x^{-1} \wedge y^{-1}$, $xy \wedge x(y \wedge z) = x(y \wedge z)$, $x \wedge yz = x \wedge y(z \wedge x^{-1}y)$.

$Q_5\{\circ, {}^{-1}, \cap, \Delta\}$ *does not contain in* $V_\infty\{\circ, {}^{-1}, \cap, \Delta\}$, *in particular,* $Q_5\{\circ, {}^{-1}, \cap, \Delta\} \neq V_5\{\circ, {}^{-1}, \cap, \Delta\}$.

THEOREM 2. $Q_4\{\circ, {}^{-1}, \cap, \cup, \Delta\} = V_4\{\circ, {}^{-1}, \cap, \cup, \Delta\}$; $V_4\{\circ, {}^{-1}, \cap, \cup, \Delta\}$ *is finitely based; an algebra* $(A, \cdot, {}^{-1}, \wedge, \vee, e)$ *belongs to* $V_4\{\circ, {}^{-1}, \cap, \cup, \Delta\}$ *if and if and only if it satisfies the equations of Theorem 1 and* $x \vee x = x$, $x \vee y = y \vee x$, $(x \vee y) \vee z = x \vee (y \vee z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

$Q_5\{\circ, {}^{-1}, \cap, \cup, \Delta\} \neq V_5\{\circ, {}^{-1}, \cap, \cup, \Delta\}$.

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