The Bounded Martin's Maximum and the Size of the Continuum

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Bounded forms of forcing axioms were introduced in [G-S]. For a class of partially ordered sets Γ, the *Bounded Forcing Axiom for* Γ, $BFA(\Gamma)$, is the following statement:

Let $\mathbb{P}$ be a poset in Γ and let $\mathcal{A}$ be a set of size at most $\aleph_1$ consisting of maximal antichains of $\mathbb{P}$, all of size at most $\aleph_1$. Then there is a $\mathcal{A}$–generic filter $G \subseteq \mathbb{P}$.

Bounded forcing axioms are arguably natural axioms, since they can be characterized in terms of a principle of generic absoluteness in the following way ([B], [M]):

Fact 1 *Let Γ be a class of complete Boolean algebras. Then the following conditions are equivalent:*

(a) $BFA(\Gamma)$.

(b) For every $a \in H(\omega_2)$ and every Σ_1 formula $\varphi(x)$, if there is a poset $\mathbb{P}$ in Γ forcing $\varphi(\check{a})$, then $H(\omega_2) \models \varphi(a)$.

The *Bounded Martin's Maximum* (BMM) is $BFA(\Gamma)$, where Γ is the class of all stationary set preserving posets.
Forcing axioms are known to decide the size of the continuum. It is therefore a natural problem to find out whether bounded forcing axioms also do.
We prove the following result:

Theorem 2 *Assume BMM and suppose $X^\sharp$ does not exist for some $X \in H(\omega_2)$. Then $2^{\aleph_0} = \aleph_2$.*

This should be compared with the following result, due to Woodin ([W]):

Theorem 3 *Assume BMM and suppose that there is a measurable cardinal or that the nonstationary ideal over ω_1 is precipitous. Then $2^{\aleph_0} = \aleph_2$.*

Also, Theorem 2 can be improved by considering natural strengthenings of the absoluteness principle in (b) of Fact 1.

[B] J. Bagaria, *Bounded forcing axioms as principles of generic absoluteness*, Archive for Mathematical Logic. To appear.

[G-S] M. Goldstern, S. Shelah, *The Bounded Proper Forcing Axiom*, J. Symbolic Logic, vol. 60 (1995), 58–73.

[W] H. Woodin, *The axiom of Determinacy, Forcing Axioms, and the Nonstationary ideal*, De Gruyter Series in Logic and its Applications. Number 1. Berlin, New York, 1999.

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